Reconstruction of the boundary conditions of heat transfer and refinement of thermophysical properties by solving inverse problems of heat conduction

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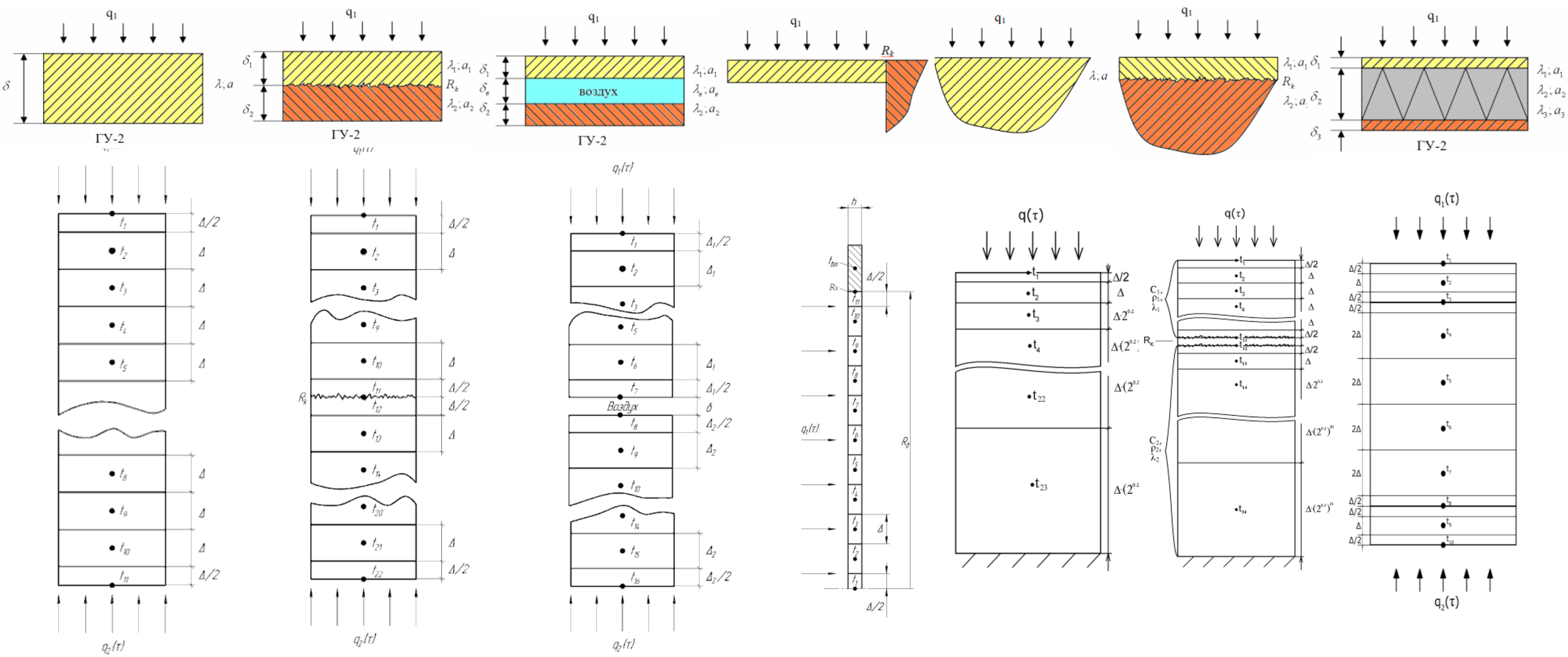
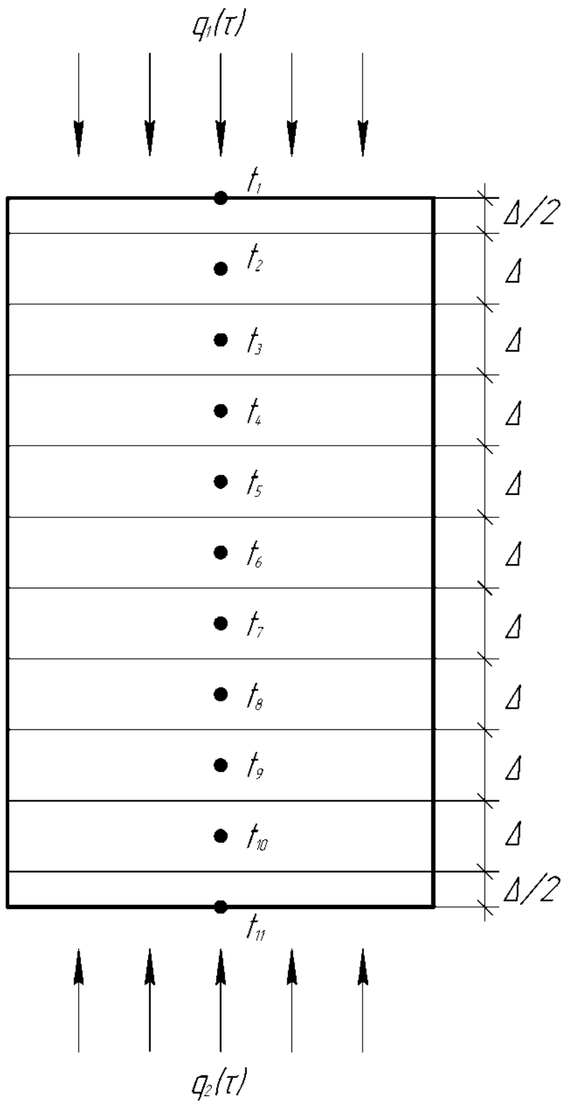
1. Schematic thermal diagrams of common heat flow detector (HFD)



2. Building a Differential-Difference Model



$$
\begin{cases}
q_1(\tau)S = C_1\dfrac{dt_1}{d\tau} + \dfrac{\lambda S}{\Delta}(t_1 - t_2)\\
\dfrac{\lambda S}{\Delta}(t_1 - t_2) = C_2\dfrac{dt_2}{d\tau} + \dfrac{\lambda S}{\Delta}(t_2 - t_3)\\
\dfrac{\lambda S}{\Delta}(t_2 - t_3) = C_3\dfrac{dt_3}{d\tau} + \dfrac{\lambda S}{\Delta}(t_3 - t_4)\\
\dfrac{\lambda S}{\Delta}(t_3 - t_4) = C_4\dfrac{dt_4}{d\tau} + \dfrac{\lambda S}{\Delta}(t_4 - t_5)\\
\dfrac{\lambda S}{\Delta}(t_4 - t_5) = C_5\dfrac{dt_5}{d\tau} + \dfrac{\lambda S}{\Delta}(t_5 - t_6)\\
\dfrac{\lambda S}{\Delta}(t_5 - t_6) = C_6\dfrac{dt_6}{d\tau} + \dfrac{\lambda S}{\Delta}(t_6 - t_7)\\
\dfrac{\lambda S}{\Delta}(t_6 - t_7) = C_7\dfrac{dt_7}{d\tau} + \dfrac{\lambda S}{\Delta}(t_7 - t_8)\\
\dfrac{\lambda S}{\Delta}(t_7 - t_8) = C_8\dfrac{dt_8}{d\tau} + \dfrac{\lambda S}{\Delta}(t_8 - t_9)\\
\dfrac{\lambda S}{\Delta}(t_8 - t_9) = C_9\dfrac{dt_9}{d\tau} + \dfrac{\lambda S}{\Delta}(t_9 - t_{10})\\
\dfrac{\lambda S}{\Delta}(t_9 - t_{10}) = C_{10}\dfrac{dt_{10}}{d\tau} + \dfrac{\lambda S}{\Delta}(t_{10} - t_{11})\\
\dfrac{\lambda S}{\Delta}(t_{10} - t_{11}) + q_2(\tau) = C_{11}\dfrac{dt_{11}}{d\tau}
\end{cases}
$$

$$
\begin{cases}
\dfrac{dt_1}{d\tau} = -2bt_1 + 2bt_2 + 2dq_1(\tau)\\
\dfrac{dt_2}{d\tau} = bt_1 - 2bt_2 + bt_3\\
\dfrac{dt_3}{d\tau} = bt_2 - 2bt_3 + bt_4\\
\dfrac{dt_4}{d\tau} = bt_3 - 2bt_4 + bt_5\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
\dfrac{dt_8}{d\tau} = bt_7 - 2bt_8 + bt_9\\
\dfrac{dt_9}{d\tau} = bt_8 - 2bt_9 + bt_{10}\\
\dfrac{dt_{10}}{d\tau} = bt_9 - 2bt_{10} + bt_{11}\\
\dfrac{dt_{11}}{d\tau} = 2bt_{10} - 2bt_{11} + 2dq_2(\tau)
\end{cases}
$$

$$b = \frac{\lambda}{C\rho\Delta^2} \qquad d = \frac{1}{C\rho\Delta}$$

3. Differential-Difference Model (DDM)

Heat transfer model in the object:

$$\frac{d}{d\tau}\vec{T}(\tau) = F\vec{T}(\tau) + G\vec{U}(\tau), \quad (1)$$

$$\vec{T}(\tau) = \begin{bmatrix} t_1(\tau) \\ t_2(\tau) \\ \vdots \\ t_{10}(\tau) \end{bmatrix}, \quad G = \begin{bmatrix} 2d & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 2d \end{bmatrix}, \quad F = \begin{bmatrix} -2b & 2b & 0 & \dots & 0 \\ b & -2b & b & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & b & -2b & b \\ 0 & \dots & 0 & 2b & -2b \end{bmatrix},$$

$$\vec{U}(\tau) = \begin{bmatrix} q_1(\tau) \\ q_2(\tau) \end{bmatrix}, \quad d = \frac{2}{c\rho\Delta}, \quad b = \frac{\lambda}{c\rho\Delta^2}.$$

Measurement model: $$\vec{Y}(\tau) = H\vec{T}(\tau) + \vec{\varepsilon}(\tau), \quad (2)$$

$H = [1 \quad 0 \quad \dots \quad 0 \quad 0]$ – measurement matrix.

4. A numerical method for solving direct problems of heat transfer in HFD based on the DDM

General solution for systems of one-dimensional differential equations (SODDE)

$$\frac{d}{d\tau}\vec{T}(\tau)=\Phi(\tau,\tau_0)\cdot\vec{T}(\tau_0)+\int_{\tau_0}^{\tau}\Phi(\tau,\theta)\cdot G(\theta)\cdot\vec{U}(\theta)d\theta \qquad \textbf{(3)}$$

For linear SODDE

$$\Phi=I+F\Delta\tau+\frac{1}{2!}F^2\Delta\tau^2+\ldots+\frac{1}{m!}F^m\Delta\tau^m+\ldots \qquad \textbf{(4)}$$

$$\vec{T}_{k+1}=\Phi\cdot\vec{T}_k+\frac{1}{2}(I+\Phi)\cdot G\vec{U}_k\Delta\tau, \qquad \textbf{(5)}$$

For nonlinear SODDE

$$\Phi_{k+1,k}=I+F_{k+1}\Delta\tau+\frac{1}{2!}F_{k+1}^2\Delta\tau^2+\ldots+\frac{1}{m!}F_{k+1}^m\Delta\tau^m+\ldots \qquad \textbf{(6)}$$

$$\vec{T}_{k+1}=\Phi_{k+1,k}\cdot\vec{T}_k+\frac{1}{2}(I+\Phi_{k+1,k})\cdot G\vec{U}_k\cdot\Delta\tau \qquad \textbf{(7)}$$

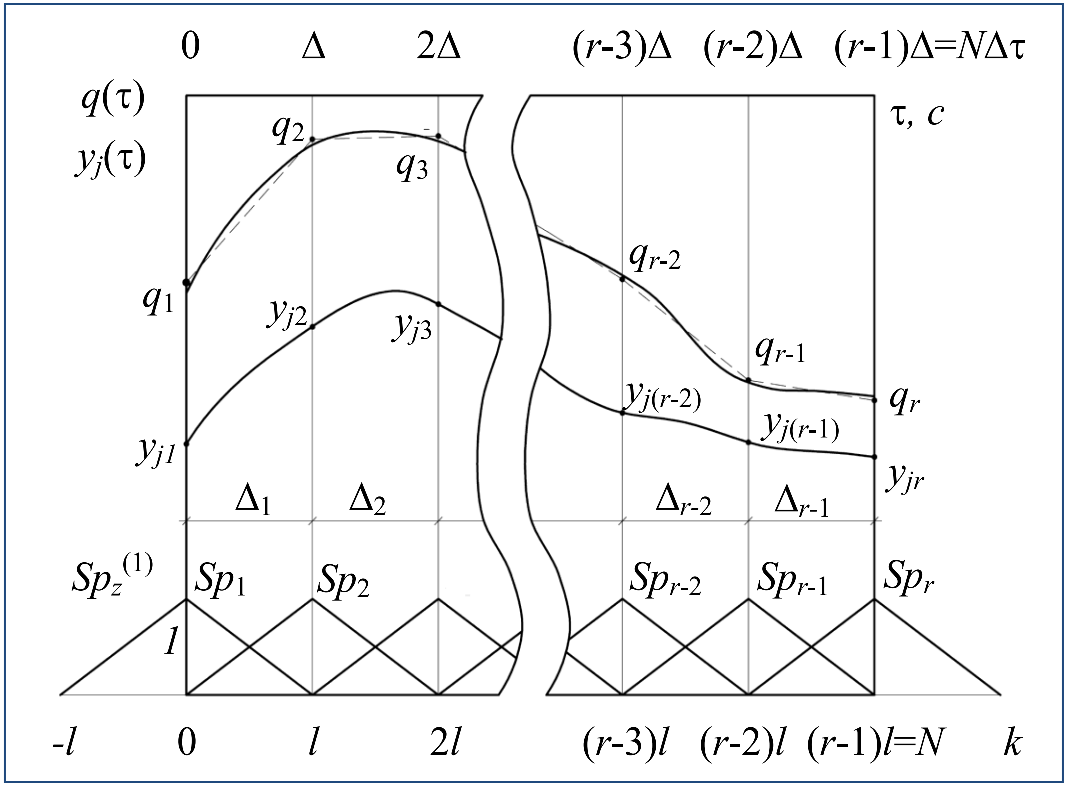
5. Parametrization of the inverse problem of heat conduction

1. Represent the required heat flux in the form:

$$q(\tau) = \sum_{z=1}^{r} q_z \varphi_z(\tau), \quad \textbf{(8)}$$

$$\vec{Q} = \left| q_1, q_2 \ldots q_z \right|^T -$$

vector of required parameters



2. Choice of first-order B-splines as basis functions

$$Sp_z^{(1)} = \begin{cases} 1 - |\xi_z|, & \text{если} \quad |\xi_z| \le 1, \\ 0, & \quad\quad\;\; |\xi_z| > 1, \end{cases} \quad \text{где} \quad \xi_z = \frac{\tau}{\Delta} - z + 1 \quad \textbf{(9)}$$

6. Solution of the Inverse problem of heat conduction

Minimizing the residual function $\Phi(\vec{Q})$:

$$\Phi(\vec{Q}) = \sum_{k=1}^{N} [\vec{Y}_k - \hat{\vec{Y}}_k(\vec{Q}_k)]^T \cdot R^{-1} [\vec{Y}_k - \hat{\vec{Y}}_k(\vec{Q}_k)], \qquad \textbf{(10)}$$

$$\underset{(m\times m)}{R} = \begin{bmatrix} \sigma_1^2 & 0 & \bullet & 0 \\ 0 & \sigma_2^2 & \bullet & 0 \\ \bullet & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \sigma_m^2 \end{bmatrix}$$

7. Using the Kalman filter for parametric identification

Nonlinear discrete algorithm Kalman filter:

$$K_{k+1} = P_k H_k^T (H_k P_k H_k^T + R)^{-1}, \quad \textbf{(11)}$$

$$\hat{\vec{Q}}_{k+1} = \hat{\vec{Q}}_k + K_{k+1}[\vec{Y}_{k+1} - \hat{\vec{Y}}_{k+1}(\vec{Q}_k)], \quad \textbf{(12)}$$

$$P_{k+1} = P_k - K_{k+1} H_k P_k, \quad \textbf{(13)}$$

$$\underset{(r\times r)}{P_k} = \begin{bmatrix} p_{11,k} & p_{12,k} & p_{13,k} & \bullet & p_{1r,k} \\ p_{21,k} & p_{22,k} & p_{23,k} & \bullet & p_{2r,k} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ p_{r1,k} & p_{r2,k} & p_{r3,k} & \bullet & p_{rr,k} \end{bmatrix}$$

$$\underset{(m\times 2)}{H_k} = \left.\frac{\partial \vec{Y}_k}{\partial \vec{Q}}\right|_{\vec{Q}=\hat{\vec{Q}}_k} = \left.\begin{vmatrix} \dfrac{\partial y_{1,k}}{\partial q_a} & \dfrac{\partial y_{1,k}}{\partial q_b} \\ \bullet & \bullet \\ \dfrac{\partial y_{m,k}}{\partial q_a} & \dfrac{\partial y_{m,k}}{\partial q_b} \end{vmatrix}\right|_{\vec{Q}=\hat{\vec{Q}}_k} = \begin{vmatrix} U_{11,k} & U_{12,k} \\ U_{21,k} & U_{22,k} \\ \bullet & \bullet \\ U_{m1,k} & U_{m2,k} \end{vmatrix}$$

8. Solution of the coefficient problem to refine the thermal conductivity

Generalized vector of parameters:

$$\vec{Q}_z = \left| \vec{Q}_{q,z} \quad \vec{Q}_{\lambda,z} \right| = \left| q_{a,z} \quad q_{b,z} \quad \lambda_z \right|^{\mathrm{T}} \tag{14}$$

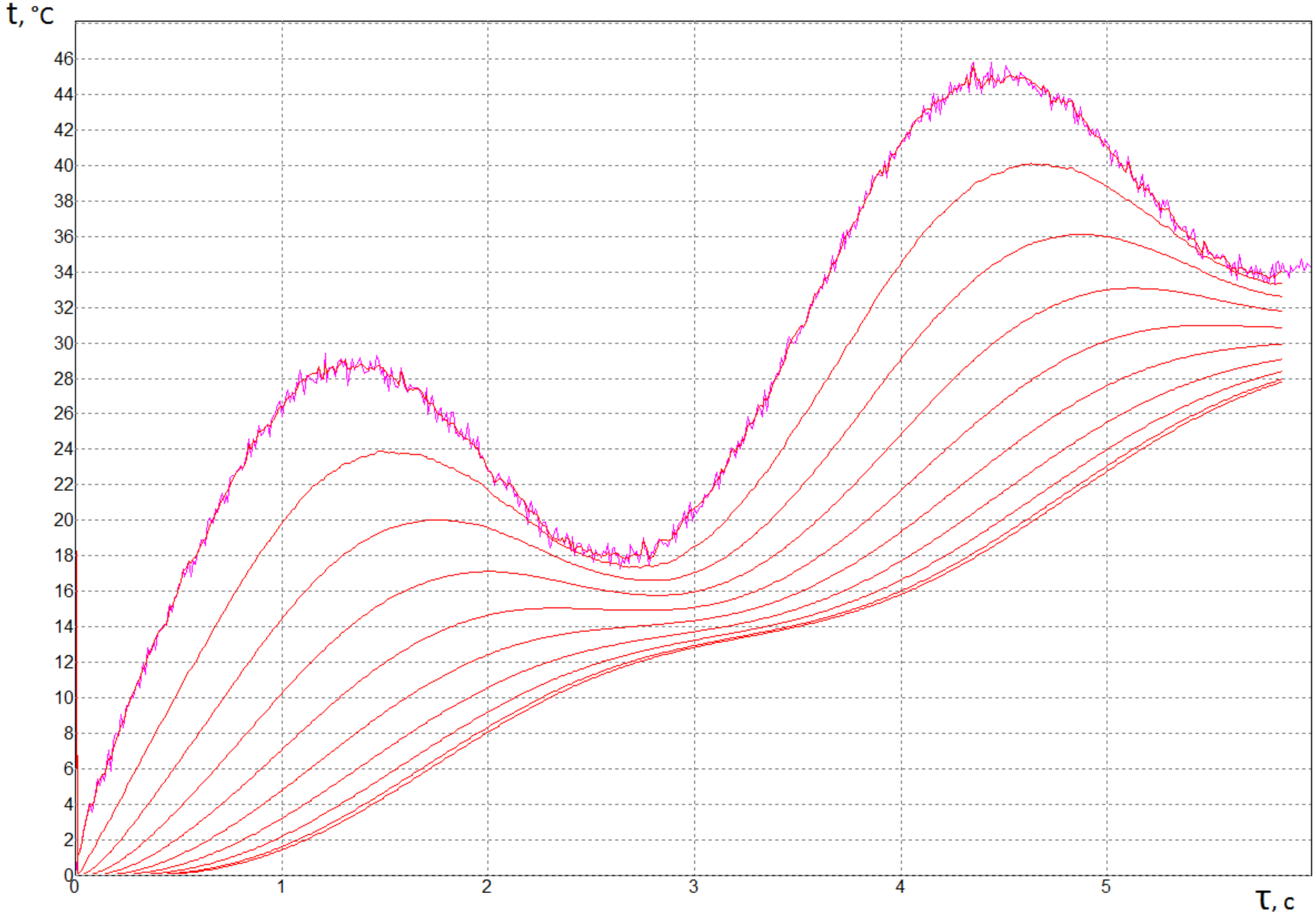
Sensitivity function matrix:

$$H_{k+1} = \left. \frac{\partial \vec{Y}_{k+1}}{\partial \vec{Q}_z} \right|_{\vec{Q}_z = \hat{\vec{Q}}_{z,k}} = \begin{bmatrix} U_{1,q_a,k+1} & U_{1,q_b,k+1} & U_{1,\lambda,k+1} \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots \\ U_{m,q_a,k+1} & U_{m,q_b,k+1} & U_{m,\lambda,k+1} \end{bmatrix}_{\vec{Q}_z = \hat{\vec{Q}}_{z,k}} \tag{15}$$

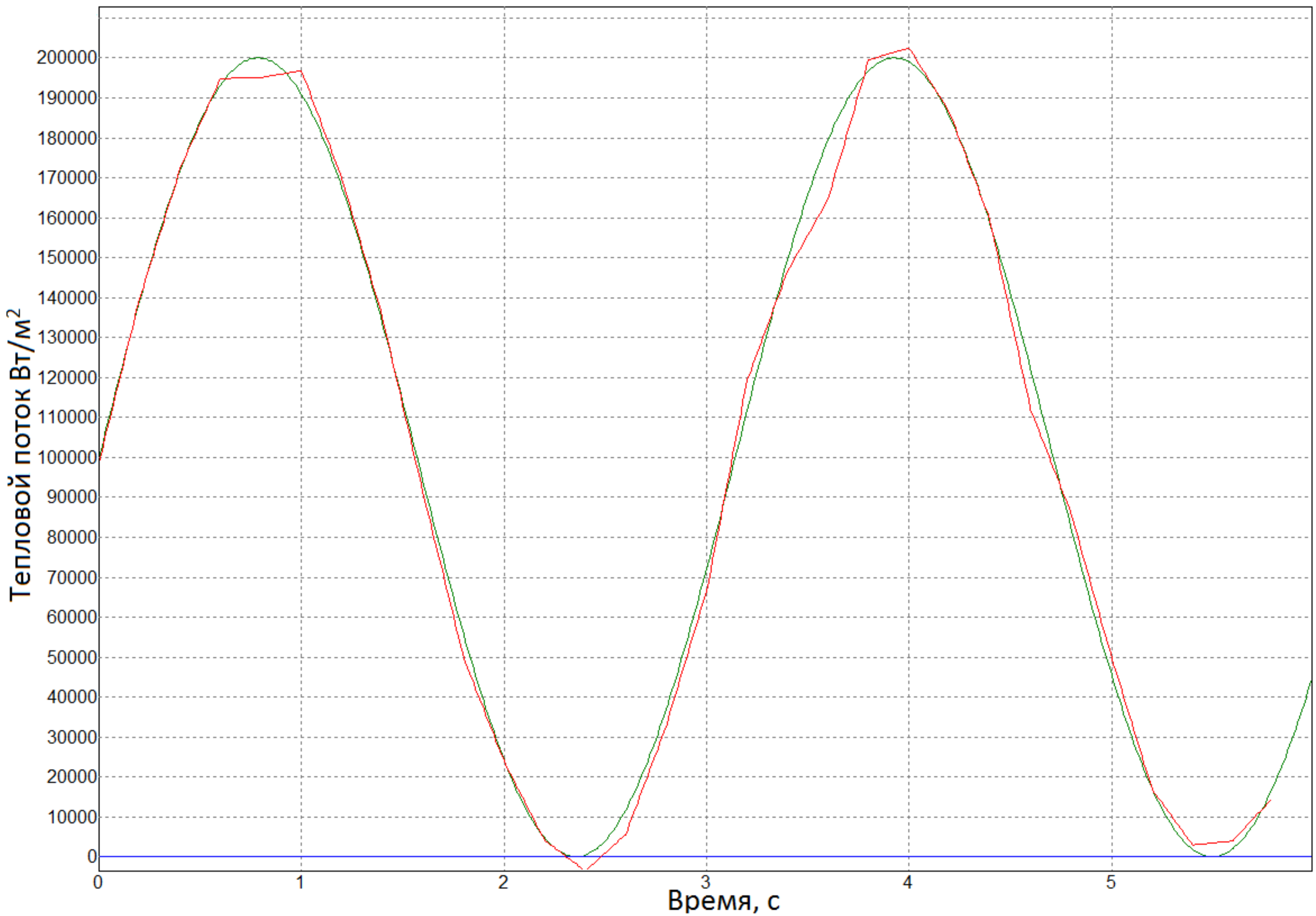
Sensitivity calculation method:

$$U_{j,\lambda,k+1} = \frac{y_{j,k+1}\left(\hat{q}_{ak}, \hat{q}_{bk}, \lambda_k \pm \Delta\lambda\right) - y_{j,k+1}\left(\hat{q}_{ak}, \hat{q}_{bk}, \lambda_k\right)}{\Delta\lambda} \tag{16}$$

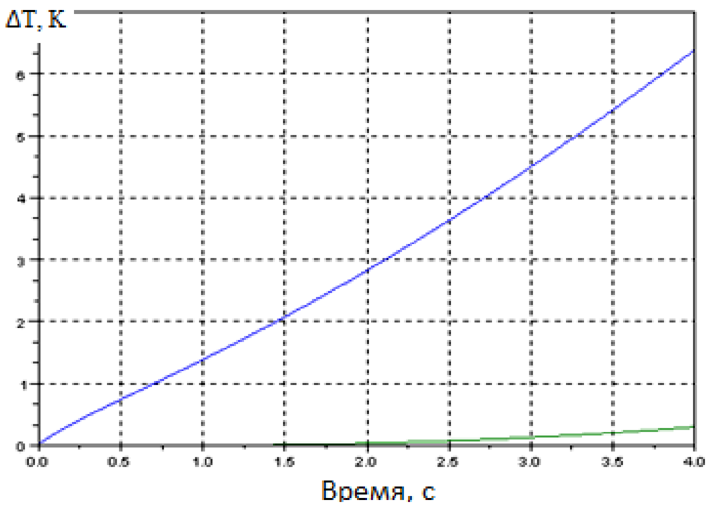
9. Temperature change in a model experiment

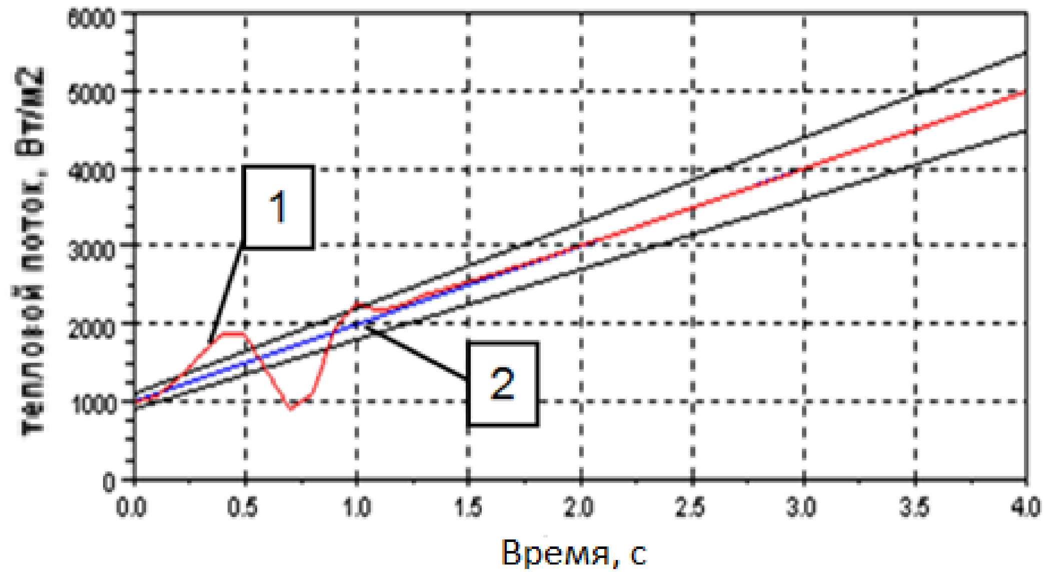


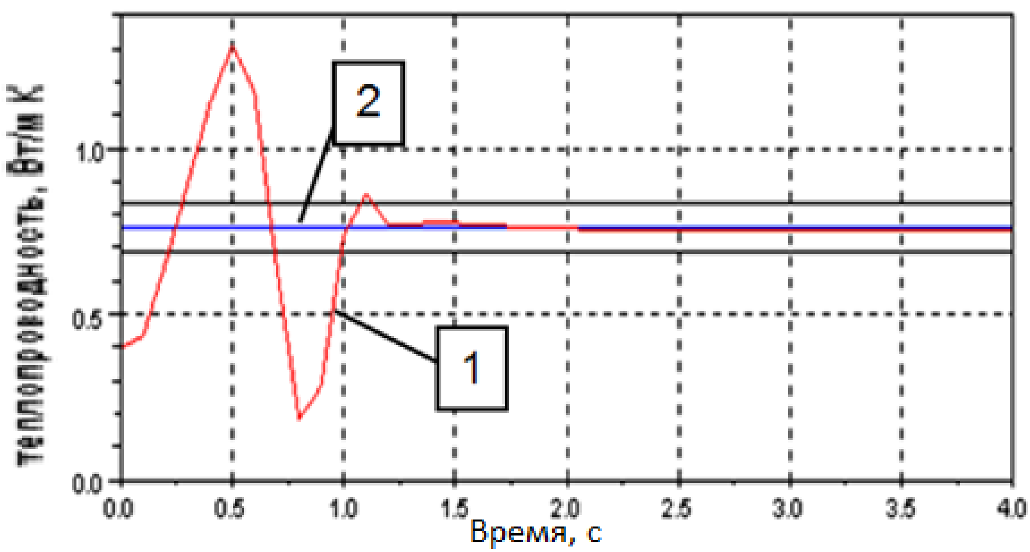
10. Heat flow change



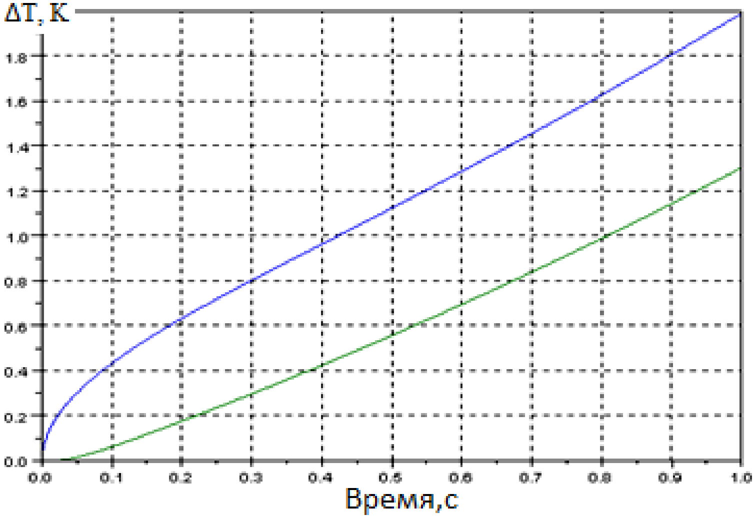
11. Simulation results (p.1)

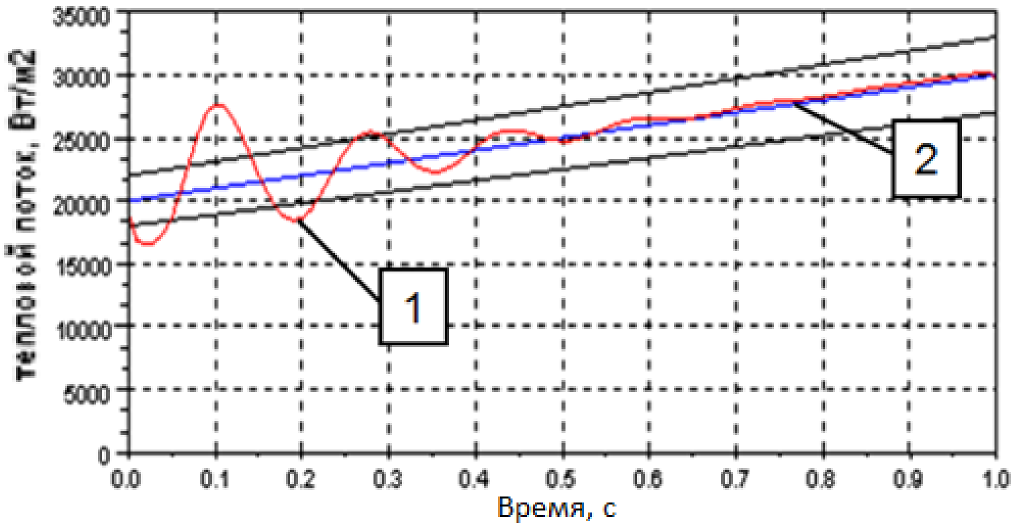


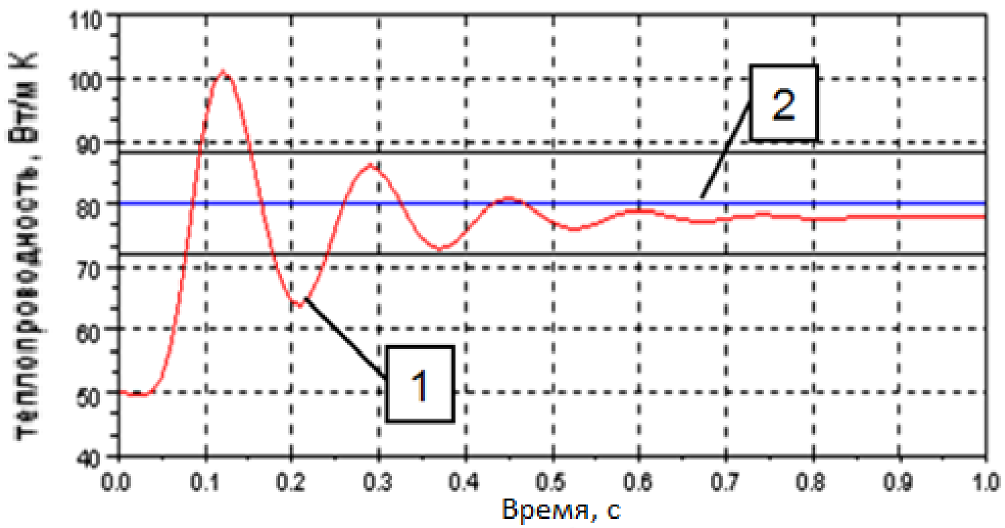




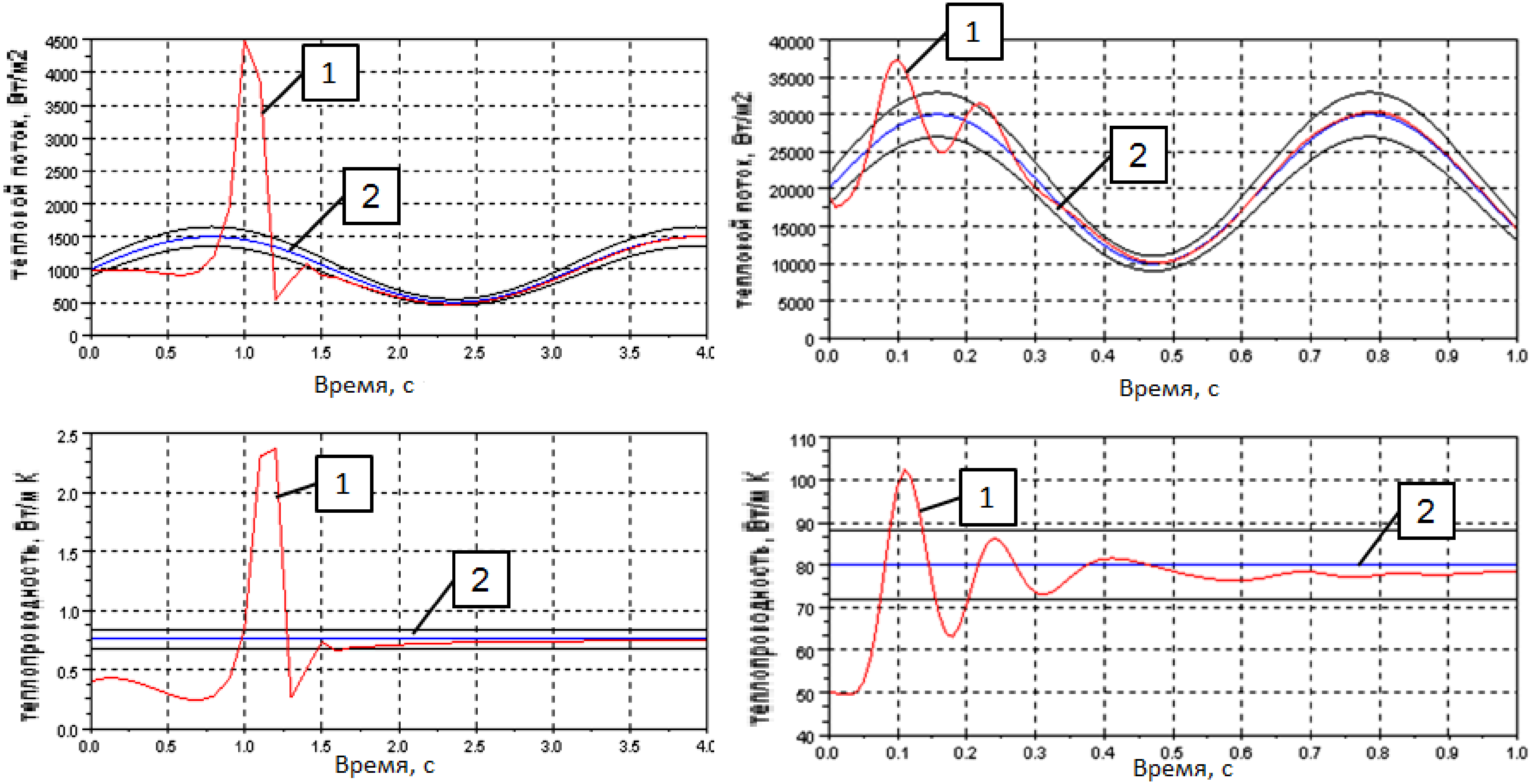
12. Simulation results (p.2)







13. Simulation results (p.3)



Conclusion

- Proposed a method and substantiated for solving the combined (boundary and coefficient) inverse problem of thermal conductivity to restore the heat flux density and simultaneously refine the thermal conductivity of the material from one experiment by changing the temperature of the object and the known boundary conditions on its rear side.
- To solve the problem, the method of parametric identification of the differential-difference model of heat transfer in the sample was used.
- The results of modeling on the restoration of the heat flux density and the refinement of thermal conductivity for various materials are presented.

Thanks for watching